In the present manuscript the Authors studied theoretically the transition between gapped and gapless states in s-wave superconductors driven by magnetic impurities. Two main result are presented: a reinterpretation of the gapped-gapless transition as a topological transition and the prediction of the enhancement of the quasiparticle thermoelectric effect related to this transition. The results are presented in a clear way and appear absent of noticeable error. However, I am not convinced that the results constitute a significant advance in the field. The topological invariant the Authors propose does not reflect in any observable quantized properties, and does not indicate the presence of a topological state of matter (unlike, e.g., Chern number in a p+ip 2D superconductor). The transition to the gapless phase, as pointed out by the Authors, does not reflect a new universality class and belongs to the same class as the Lifshitz transition. Moreover, thermoelectric effect has been investigated in superconductors with magnetic impurities previously (Refs. 32,32) using various approximations; the Authors do not make it clear how their approach is different and if their results offer anything new except for the discussion. Finally, there are also questions regarding the generality of results (for example, whether the same results will hold if the gap is not isotropic) and the approximations used (in particular, the Born approximation appears to be implied).

Consequently, I do not recommend the manuscript for publication in SciPost Physics. Below I expand on the points mentioned above and provide questions and suggestions to be addressed in the manuscript.

(1) The Authors offer a reinterpretation of this transition in terms of a topological invariant - the Euler characteristic calculated for the DOS surface. Are any physical quantities of the system uniquely determined by this invariant?

• Is there any interplay of the Euler characteristic with the known cases of topology in superconductors (e.g. p+ip state in 2D)? Being a property of DOS, I believe that it will not distinguish between trivial (s-wave) and topological (p+ip in 2D) states.

(2) All calculations were performed for the case of an ideally isotropic gap. How will gap anisotropy affect the results? In particular, for strong anisotropy with deep minima, can the behavior of DOS and free energy near the transition change qualitatively?

(3) The Authors refer to Abrikosov-Gor'kov theory for the disordered superconductivity throughout the text - does that imply that Born approximation for scattering is used? In particular, does the justification for the stability of the mean-field description given in "Smearing of the transition due to spacial fluctuations of the magnetic impurities concentration", rely on Born approximation? How will rare region effects and the presence of impurity bound states affect the argument?

(4) The relation to previous works on thermoelectric coefficients in superconductors with magnetic impurities has to be discussed in more detail. In Ref. 33, multiple scattering effects were considered beyond the Born approximation, so it appears that the results of Ref. 33 are more general. In Ref. 32, on the other hand, results in Born approximation were reported and Eq. (11) and (12) there are indeed quite similar to Eq. (13) and (8) of the current work. However, the denominators in Eq. (13) and Eq. (11) of Ref. 32 appear different - could the Authors explain this difference?

• Since results at finite T are reported in Fig. 3, was $\Delta(\zeta)$ calculated self-consistently for finite T or were the zero-temperature expressions used? If the latter is true, this has to be mentioned and justified.

Minor comments/suggestions:

— Fig. 1 misses a color scale; the drastic change from purple to blue seems to suggest a jump, which can be confusing to readers, since the transition is actually continuous.

— Mentions of the applications to $s_{++}/s_{\pm}$ transition and color superconductivity in QCD and string theory are not really substantiated or discussed. The Authors should either provide a discussion of what new physics can their approach reveal in those systems or refrain from stating the connection (at least in the abstract and conclusion).

— Some links in citations are not working (e.g. 17,18); Ref. 17 links to the same URL as Ref. 18; Ref. 18 is missing the journal information.