REPORT:

HarmonicBalance.jl: A Julia suite for nonlinear dynamics using harmonic balance.

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The authors present and discuss an open-source software package that allows to analyze nonlinearly coupled driven resonators.

Specifically, this package aims at offering a tool for analyzing the steady state of the oscillations using the method of the Harmonic Balance. The latter method allows, provided a suitable ansatz, to reduce the problem of the time-dependent equations of motion of N driven nonlinear resonators to the problem of nonlinearly coupled equations for the amplitudes (with first derivative in time). This method is well known and widely used by the community working in this field.

The paper is well written and it is a good pedagogical introduction to the topic of nonlinear coupled resonators.

At the introduction, the authors give a narrow overview of the literature who tackles the problems of nonlinear resonators and missed books and relevant references, as for example:
- A. H. Nayfeh and D. T. Mook, Nonlinear Oscillations (2008);
- H.G. Schuster, Reviews of Nonlinear Dynamics and Complexity (2009);
- M.I. Dykman, "Theory of nonlinear nonequilibrium oscillators interacting with a medium", Zh. Eksp. Teor. Fiz. 68, 2082 (1975) [Soviet Physics JETP 41, 1042 (1975)];

and some recent and relevant works as for example:
- PloS One 9, e0162365 (2016);
- Appl. Phys. Lett. 114, 103103 (2019);
- Phys. Rev. Applied 13, 014049 (2020);
- Nature Nanotechnology 12, 631 (2017);
- Nature Communications 8, ncomms15523 (2017).

In some interesting cases, the steady state solution can correspond to limiting cycles of the slow varying amplitude, see for example: Phys. Rev. Lett. 118, 033903 (2017), Phys. Rev. Lett. 121, 244302 (2018), Phys. Rev. Lett. 122, 254301 (2019, generating frequency comb. In this case, these variables are not described by fixed (time independent) points but they can have very different frequencies from the drive (not commensurate and different in scale). This possibility is not covered by the present package which makes uses of the ansatz based on the Harmonic Balance.

Another interesting case is the subharmonic resonance response. For a single resonator, this occurs when the drive frequency is a close to a multiple of the bare frequency. For example, for the Duffing resonator, the drive frequency can be 3 times the harmonic frequency. In this case, the system can oscillate to a fraction of the drive frequency and, this case cannot be addressed by the package, as I understood.

The two effects mentioned are theoretically interesting as they are due to the breaking of the discrete time translational symmetry on which is based the Harmonic Balance method.

Since it is not discussed, I assumed that the package cannot address the problem of the ring-down dynamics of the nonlinear resonators which is an important and relevant problem in the field. The ring-down measurement is valuable experimental way to reveal nonlinear effects affecting the system, see for instance for example Scientific Reports 7, 18091 (2017).

It is also worthy to mention the switching processes induced by the noise between the stable solutions that generates a slow dynamics that is not related to the drive frequency, see for example: Phys. Rev. Lett. 65, 48 (1990), Phys. Rev. E 49, 1198 (1994), Phys. Rev. E 57, 5202 (1998), Phys. Rev. E 74, 061118 (2006), Phys. Rev. Lett. 100, 130602 (2008), Phys. Rev. E 92 050903 (2015). But I understand that the inclusion of the noise is beyond the scope of the present software as well as the possibility to calculate theoretically the power spectrum which gives essential information about the nonlinear systems, see for instance Nature Nanotechnology 11, 552 (2016), and Nature Communication 5, 5819 (2014).

In this sense, the scope of this package looks quite modest as it does not address essential aspects of the nonlinear physics.

But the most critical question is the following: for which experimental systems this tool can be useful? I mean the following.

The authors report the example of two linearly coupled Duffing resonators and I agree that a such kind of system can represent many experimental systems, from nanomechanics to optomechanics, and so on. Indeed, this problem has been studied theoretically by many theoreticians, at least for the fundamental harmonics. Naturally, one can analyze the role of other harmonics and this is, in some sense, equivalent to increase the number of harmonic components.

The authors present a table with maximum N=5 resonators. From my knowledge, most of the experiments studying nonlinear resonators are confined to investigate a very limited numbers of coupled modes and the majority of these studies focus on a few modes (N=2 or N=3). When we increase this number, one has to face a “zoo” of solutions from which it is in general difficult to gain some physical insight, unless one wants to merely compare theoretical points with experimental data.

On the other hand, if one increases the size (for example a long chain of couple Duffing resonator with tens of resonators) then one has to solve a fully many-body (classical) problem. In this context, the method presented here should be compared with other more sophisticated techniques as Molecular Dynamics and must demonstrate its advantages. Furthermore, when we have a long chain, one could attack the problem from a different point of view, namely by using the linear modes of the coupled particles: the chain’s modes.

In this case, I expect that the effective nonlinear coupling between the several chain's modes is only relevant when we have internal resonances between these modes and this should restrict, again, the study to a few modes.

In other words, from a point of view of basic research (and not for engineering) it is not clear to me what is the fascinating and new physics that would emerge by analyzing an increasing number of resonators or only two modes but with many harmonics.

Of course, this package offers a useful tool for having (in a fast time) solutions for problems who have been already addressed and investigated theoretically and experimentally. Therefore I am not fully convinced for its publication.