REPORT ON "FAMILIES OF SOLUTIONS OF THE HETEROTIC G_2 SYSTEM"

The heterotic equations of motion have been studied for many years in theoretical physics and, in the form of the Hull-Strominger system, have provided an interesting geometric structure in 3-dimensional complex geometry that explores non-Kähler analogues of the Calabi-Yau condition. In dimensions other than 6, the study has been more sporadic. Results have been given on parallel spinors, connections with skew-torsion, instantons in higher dimensions and relations to Lie and Courant algebroids, but only gradually is the meaning of the full heterotic system being understood. Perhaps for physical reasons or perhaps for the connection to Riemannian holonomy, the heterotic system in 7-dimensions is now one of the most important cases, providing an interesting torsion-class of G_2-structures and benefitting from previous studies on G_2-gauge theory.

The heterotic G_2-system on a 7-manifold X consists of a G_2-structure on X satisfying a certain torsion condition, an instanton on the tangent bundle and an instanton on some other auxiliary bundle, together satisfying the so-called Bianchi identity relating the torsion of the G_2-structure with a quadratic expression in the curvatures of the instantons.

The current paper is an extensive study of a very natural set of new examples. The authors consider homogeous 7-dimensional squashed 3-Sasaki manifolds, those being members of a 1-parameter family of G_2-structures coming the dilation of the fibre of a Riemannian submersion. I find the terminology somewhat imprecise, given that only for $s = 1$ is the metric 3-Sasakian. The homogeneity hypothesis firstly restricts to only two examples (the 7-sphere and an Aloff-Wallach space) and also determines a distinguished principal bundle that can be used to construct the instantons. The authors consider three classes of connections, two of which are defined on the principal bundle and one only on the tangent bundle.

The principal result of the paper is the construction of many explicit solutions to the heterotic G_2 system, by an analysis of the nine different possible pairs of instantons (tangent and gauge) and how they contribute to the Bianchi identity. Some cases are excluded, according to the necessity for the string parameter α' to be positive. However several cases lead 1-parameter families of solutions. The authors also note how α' varies with s, commenting on the physical relevance of the solutions.

The construction of new solutions to this interesting and difficult system of equations is very welcome, and gives new insight to questions of moduli of solutions.

The article is generally well written, though some imprecise terminology is used (as noted below), especially in the introduction. However, in addition to the stated results, the paper gives a thorough and welcome introduction to the literature in this quickly growing field.

- p. 5 (final parag.) I would say that the Killing fields ξ_i *generate* an integrable distribution.
- p. 6 (first parag.) The triple $\{\xi_i\}$ can be extended to a local frame satisfying Equations (2.2), (2.3), but it is not true that these equations are satisfied for any such local frame. See, for example, [BG, Thm. 3.3.4].
- p. 7 (Equation (2.11)) It should be noted that the 2-forms ω^k commute with the η^i (while the η^i and η^j anti-commute). I am not sure that this is taken account of in the ϵ_{ijk}-notation.
- p. 8 (fourth parag.) It is not clear how the Maurer-Cartan form defines a metric on G. The basis $\{e^\alpha\}$ is said to be orthonormal, but in the previous paragraph it is taken to be an arbitrary basis.

- p. 8 (fifth parag.) Instead of the term *coset space* or *coset* I much prefer the more standard term *homogeneous space*. Use of the word coset, here and later, to describe G/H is extremely confusing. $\mathfrak{m}$ is not a Lie subalgebra. It is an $\mathfrak{h}$-module.
- p. 14 (second sentence) This is imprecise. The vector bundle of self-dual forms is the vector bundle associated to the principal $SU(2)$-bundle, via a representation on $\mathbb{R}^3$.
- p. 18 (tenth line) Again, change terminology to say that the homogeneous space G/H is reductive. The representation is given by the adjoint representation on $\mathfrak{m} \subseteq \mathfrak{g}$.
- p. 25 (tenth line) It is not clear what range is being discused. Is it the range of s-values, or a-values or c?
- p. 26 Throughout Section 5, solutions are listed as "Solution 1, Solution 2, etc" but no attempt is made to distinguish between the solutions or identify the solutions. Is it the case that for a given value of s, which may be restricted according to the "ranges" denoted above, the three solutions corespond to the roots $a(s)$ of a cubic equation in a? In the various tables, Solutions 1, 2 and 3 are listed, but with no way of distinguishing between them. Is Solution 1 that one corresponding to smallest $a(s)$ value?
- p. 36 (third parag.) This discussion does not quite show it, but this paragraph suggests the non-connectedness of moduli of solutions to the heterotic G_2 system.

REFERENCES

[BG] Charles P. Boyer and Krzysztof Galicki, 3-*Sasakian Manifolds*, Surveys in differential geometry, 6(1):123-184 (2000).