Referee report on the paper "The threefold way to quantum periods: WKB, TBA equations and q-Painlevé" by Fabrizio Del Monte and Pietro Longhi

February 27, 2023

Dear Authors, dear Editor,

It is an interesting paper that studies the relation between three different stories related to 5d supersymmetric gauge theory. The main example considered in the paper is so-called fine-tuned stratum, which corresponds to the algebraic solution of the q-Painlevé III_3 equation. In this case almost everything is computed explicitly. Then the Authors study some deformations of this case.

I think that paper is interesting and definitely deserves to be published after some not very big revision.

There are the following questions and comments:

1. There is an obvious typo in the formula (5.21): it should be uq^n instead of u.

2. Formulas (2.18) can be proved explicitly. It's a bit strange to use numerics for the computations of such kind. The idea of the proof is the following:

   - Curve (2.2) has elliptic parameterization of the following form (see, e.g., Example 3.6 in 1804.10145):

   $$e^x = -\frac{\theta_1(Z,q)\theta_1(U+Z,q)}{\theta_3(Z,q)\theta_3(U+Z,q)}, \qquad e^y = \frac{\theta_1(U+Z,q)\theta_3(Z,q)}{\theta_1(Z,q)\theta_3(U+Z,q)}, \tag{1}$$

   $$\tau = -i\frac{\theta_1(U,q)^2}{\theta_3(U,q)^2}, \qquad \kappa = 2\frac{\theta_3(q)^2}{\theta_2(q)\theta_4(q)}\frac{\theta_2(U,q)\theta_4(U,q)}{\theta_3(U,q)^2}. \tag{2}$$

   - Condition $\kappa = 0$ can be satisfied by putting $U = 1/2$. After this elliptic parameterization acquires the form

   $$e^{x(Z,q)} = i\frac{\theta_1(Z,q)\theta_2(Z,q)}{\theta_3(Z,q)\theta_4(Z,q)}, \qquad e^{y(Z,q)} = \frac{\theta_2(Z,q)\theta_3(Z,q)}{\theta_1(Z,q)\theta_4(Z,q)}, \qquad \tau(q) = -i\frac{\theta_2(q)^2}{\theta_4(q)^2}. \tag{3}$$

   - We notice the following relations

   $$e^{x(Z+1/2,q)} = -e^{x(Z,q)}, \qquad e^{y(Z+1/2,q)} = -e^{-y(Z,q)} \tag{4}$$

   and

   $$e^{x(Z+\omega/2,q)} = -e^{-x(Z,q)}, \qquad e^{y(Z+\omega/2,q)} = -e^{y(Z,q)}, \tag{5}$$

   where ω stands for period of the elliptic curve.

- Using this relation we can see that the differential ydx on the two neighboring A- or B-cycles, shifted by half-period, transforms like $ydx \mapsto -ydx + \ldots$. From the other side, the difference of the two integrals over these shifted cycles (which more or less equals to 2*period) can be computed as a sum of two residues. Therefore, we are able to express the two periods of ydx in terms of some residues, which can be computed easily.
- It also seems that everything can be done even without this explicit elliptic parameterization. Namely, we can use symmetries like (4), (5), check that they preserve the branch points, and also reduce period computation to the computation of residues.

3. Solution of the q-Pianlevé equation is usually defined up to some arbitrary q-periodic functions of t. In particular, we know that there is a 1-parametric family of solutions given by the spectral determinant. It is different from the naive $s = 1$ limit in (5.21), and the difference is something like NS free energy with $\hbar \mapsto \frac{2\pi}{\hbar}$.

   So the question is, how does one know that the actual solution of the TBA equation is (5.21), and not, for example, this spectral determinant? Maybe it is even better to ask first what is possible ambiguity in the solution of the TBA equations, and how it related to the ambiguity on the Painlevé side. And probably there is the same question about the WKB side, what is the ambiguity there.

   Anyway, it seems that some precise statement about the relation between TBA and the q-Painlevé solution is missing.

4. What is the relation of the spectral determinant mentioned above to this threefold story? Is it related somehow to this 1-parametric ρ deformation, or it is some different direction?