Referee report

The authors have studied the correspondence between the Q-systems for integrable spin chains of $A_{\ell-1}$ type and 3d $\mathcal{N}=4$ quiver gauge theories known as $T_\rho^\sigma[SU(N)]$. They establish a dictionary which connects some features in the gauge theory side (such as Coulomb and Higgs branch higgsings, and mirror symmetry) to correspondent operations in the Q-system side.

I regard this work as extremely interesting, and as an example of high level research in the field. At the practical level, the paper is also clear and well written, and I could not find typos. I have just some minor comments and questions I would like to ask to the authors.

- It is claimed various times in the text that is it much faster to solve for the Bethe roots by using the Q-system instead of the Bethe ansatz equation. This is summarized in table 1. It is not clear to me if these values refer to the XXX or XXY model. Also, the first two entries of the table are numbers in the same order of magnitude, so I personally don't know if one can use those two data points to argue for the superiority of the rational Q-system. It would be good if the authors could expand a little this section.

- In section 2.2 it is assumed various times that some terms appearing in eq. (2.23) do not vanish, in order to divide the expression by them and have them appear in denominator. I don't understand how strong this assumption is, or if anything can be said in the vanishing case.

- In various places in the text it is claimed that the 3d mirror dual of $T_\rho^\sigma[G]$ is given by $T_\sigma^\rho[G^\vee]$. I believe there are subtleties with this statement, which were overlooked in the original papers on $T_\rho^\sigma[G]$ theories by Hanany et. al. In particular for G orthogonal or symplectic, there exist consistent choices of ρ and σ such that $T_\rho^\sigma[G]$ is a bad theory (here bad, in the sense of Gaiotto-Witten). As an example one can realize $USp(4)-[SO(8)]$ in this way. Now, for bad theories the concept of mirror symmetry is not unique. The Coulomb branch is not a cone with a unique singular point at which the SCFT and its mirror live at low energy. In general there are various singular higher dimensional loci, and one can define a different 3d mirror for each one of them. I was thinking maybe the authors could add this comment in a footnote somewhere in section 7. Anyways this comment is irrelevant for the rest of the paper as for G special unitary, any choice of partitions leads to good theories.

Upon clarification of the small points above, I will be happy to recommend this paper for publication on Scipost. I further add some other personal questions I have, which don't necessarily need to be addressed. It is just out of my personal interest for the paper.

1. In the case G is orthosymplectic, does anything unusual or interesting happen in the Q-system side, in the case in which ρ and/or σ are orbits which do not lie in the image of the Spaltestein map?

2. Also in the orthogonal case, can the Q-system distinguish cases in which ρ or σ is doubly even? (i.e. the red/blue orbits for $so(4)$).

3. Is there any understanding of the field theory operators (like monopole and mesons) in the Q-system side, and how they get exchanged by mirror symmetry?

4. By glueing various $T_\rho[SU(N)]$ theories (for trivial σ) one can produce trinions, or in general star shaped quivers which are mirrors of class-S theories. Is there any way to extend this Bethe/Gauge correspondence to such theories?

5. Mirror symmetry is realized as the S-transformation in an Hanany-Witten setup. One can however use a more generic element of the IIB duality group, and have a much larger set of dualities. Like for example doing STS (where here T is the T generator of $SL(2,Z)$). This generates linear quivers with CS levels at the nodes, as now (p,q) branes with both p and q non-trivial are present in the HW cartoon. One can of course check (i.e. HS, etc) that this theory is equivalent to the original one. Can you implement this duality at the level of the Q-system?