REFEREE REPORT

Article: Straightening Out the Frobenius-Schur Indicator
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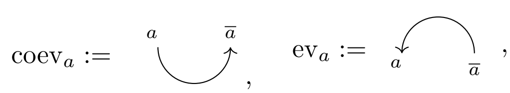
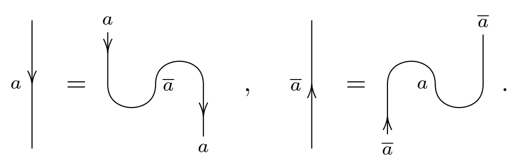
The paper under review is an interesting paper but the reviewer cannot agree that straightening out a "zig-zag" label by an object a in any fusion category yields the second indicator of a (Fig. 2). In any tensor category with duality, one can always straightening out a "zig-zag" (without any sign)! This is why duality is built into the definition of fusion categories. There seems to be some misunderstanding of the notion of left and right duals of tensor categories throughout this paper. The notion of flags of a cup (or cap) introduced in the paper is simply the left and the right dual of an object a. These concepts will be elaborated in more detail in this report.

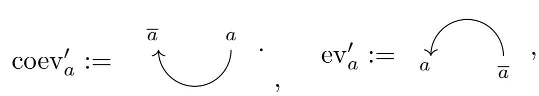
Frobenius-Schur indicators are well-defined for any fusion category equipped with a *spherical structure* $j : a \to \overline{\overline{a}}$, often called a spherical fusion category. In the case of a unitary fusion category, it admits a canonical spherical structure which yields positive dimension d_a for each simple object (or anyon) a. In general, up to equivalence, one may assume the underlying fusion category is *strict* and j is the identity. In such a strict category $\mathcal{C}$, the left dual of an object a is a triple $(\mathrm{coev}_a, \mathrm{ev}_a, \overline{a})$, where $\overline{a}$ is an object in $\mathcal{C}$, called the dual of a, $\mathrm{coev}_a : \mathbb{1} \to a \otimes \overline{a}$ and $\mathrm{ev}_a : \overline{a} \otimes a \to \mathbb{1}$ are morphisms in $\mathcal{C}$. The diagrammatic notations for these morphisms are



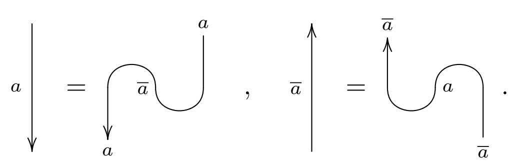
and they satisfy the "zig-zag" equations



The right dual of an object a is a triple $(\mathrm{coev}'_a, \mathrm{ev}'_a, \overline{a})$ where $\mathrm{coev}'_a : \mathbb{1} \to \overline{a} \otimes a$ and $\mathrm{ev}'_a : a \otimes \overline{a} \to \mathbb{1}$ are morphisms in $\mathcal{C}$ denoted by the diagrams

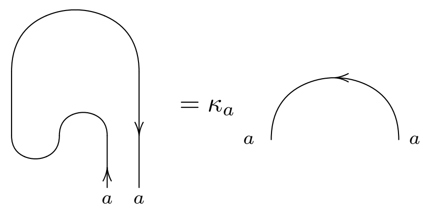


satisfying

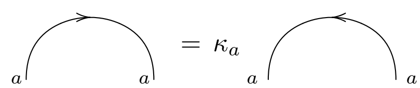


By the definition of unitary fusion categories, one can always straightening out a "zig-zag", and the second indicator has nothing to do with straightening out a "zig-zag"!

Using the mathematical definition of the second Frobenius-Schur indicator in literature, κ_a of a self-dual anyon a is given by



After straightening out the "zig-zag" of the diagram on the left side, one can find that the equation



which means the left dual and the right dual of a are differed by the second indicator of a. In other words, changing the orientation of a cap (or a cup) will yield the scalar κ_a. The *flag* introduced in the paper is simply the orientation given by the left or right dual of a.

The article does not emphasize $N_a^{bc} \leq 1$ but Equation (1) holds only under such assumption. The gauge changes would be a lot more complicated if $N_a^{bc} \geq 1$.

In the last paragraph on page 15, it is not true that braided unitary theory has a unique ribbon structure. In a modular tensor category, the number of such ribbon structure is equal to the number of simple currents. However, the spherical structure determined by a unitary braided fusion category also canonically determines *a* ribbon structure.

Finally, the reviewer would like to comment on the notion of $\mathbb{Z}_2$ Frobenius-Schur grading of fusion algebra introduced in this paper. This notion is built on the positivity conjecture of Frobenius-Schur indicators in RCFT, which was proven to be false more than two decades ago. The corrected version of *positivity conjecture* was proven in the reference 51. One can always investigate those spherical fusion categories which satisfy the *positivity conjecture*. Obviously, the Dijkgraaf-Witten theories of odd order groups satisfy this conjecture trivially. Are there any nontrivial family of examples of such categories which naturally admit a $\mathbb{Z}_2$ Frobenius-Schur grading? What are the possible $\mathbb{Z}_2$ Frobenius-Schur grading of Dijkgraaf-Witten theories of odd order groups? Note that $\kappa_a = 0$ if $a \neq \mathbb{1}$ in these Dijkgraaf-Witten theories.

In conclusion, the article provides a confused understanding of tensor categories, duality, Frobenius-Schur indicators and their applications in TQFT. Therefore, the reviewer regretfully declines to recommend the article for publication in its current form.