REPORT ON "THE TOWER OF KONTSEVICH DEFORMATIONS FOR NAMBU-POISSON STRUCTURES ON $\mathbb{R}^d$: DIMENSION-SPECIFIC MICRO-GRAPH CALCULUS"

1. Context

Graph complexes have been introduced by M. Kontsevich in the formulation of his Formality conjecture [3] in an attempt to prove the formality of the Hochschild dg Lie algebra of multi-differential operators by showing the rigidity (in the stable setting) of its cohomology, namely the Lie algebra of polyvector fields. As a by-product, Kontsevich introduced a map from cocycles in a particular graph complex to flows on the space of Poisson manifolds. Kontsevich's note also includes an explicit formula [later corrected in [1]] for the infinitesimal flow induced by the tetrahedron graph γ_3, the simplest of an infinite family of wheel graphs γ_{2p+1}, with $p \geq 1$, being non-trivial cocycles for the corresponding graph complex [6]. Kontsevich's construction is universal *i.e.* is independent of the (affine) Poisson manifold and works in arbitrary (finite) dimension.

2. This paper

The present paper aims at studying infinitesimal flows generated by wheel cocycles (most prominently γ_3) on a sub-family of Poisson brackets, namely (a generalisation of) the Nambu-Poisson brackets. The manuscript builds on previous works of the authors, most notably [2] to which the present paper can be seen as a follow-up. In particular, it was shown in [2] that the – generically non-trivial – infinitesimal flow induced by γ_3 trivialises for Nambu-Poisson brackets in $\mathbb{R}^3$. This was proved by exhibiting an explicit expression for the trivialising vector field $X_3^{\gamma_3}$, thus constructively showing that the associated infinitesimal flow is a Poisson coboundary (for this particular class and dimension). The present paper intends to complete this result by first showing that the dimensional reduction of the trivialising vector field $X_3^{\gamma_3}$ for the γ_3-induced flow in $\mathbb{R}^3$ coincides with the known "sunflower" trivialising vector field $X_2^{\gamma_3}$ in $\mathbb{R}^2$ [Proposition 1].[1] The authors then proceed by displaying an expression for $X_3^{\gamma_3}$ being manifestly $\mathfrak{gl}(3)$-invariant [or in the words of the authors, expressed in terms of micro-graphs] and by showing that the presence of tadpoles is a necessary requirement for the trivialisation to occur [Proposition 3].

3. Opinion

Here are some remarks regarding the results presented in this work.

(1) The introduction does not sufficiently emphasise why flows should be considered as interesting in general. The authors refer to their previous paper [2] for motivation, but even there the question of the applications – or more broadly motivation – is not satisfyingly addressed. Although we agree that the proposed problem is well-posed, it will be more rewarding for the reader to go through the intricacies of the [largely brute-force] computations performed to have some minimal motivation to do so. Incidentally, the fact that 8 out of 12 citations refer to previous works of the authors [the 4 remaining citations referring to classic works not mentioning explicitly the problem treated] does not help to consider the objectives pursued in the present work as a crucially meaningful addition to the field of deformation quantization or Poisson geometry.

[1] Recall that any bivector in dimension $d = 2$ is Poisson and in particular is Nambu-Poisson.

(2) The conclusion of the paper presents a conjecture, stating in particular that the flow induced by the tetrahedral graph γ_3 is trivial (*i.e.* the induced infinitesimal deformation is a Poisson coboundary) for any Nambu-Poisson bracket in dimension $d \geq 3$. The present paper (together with its companion [2]) can be understood as providing some motivation for this conjecture by showing that the latter holds for $d = 3$. However, one can wonder what justifies the restriction to γ_3 which is not privileged (apart from being the simplest) among the wheel generators of the Grothendieck-Teichmüller algebra $\mathfrak{grt}_1$ $\big[$isomorphic to the zeroth cohomology of the even graph complex [6]$\big]$. A more sensible version of the conjecture would thus read "the flow induced by the wheel generators of $\mathfrak{grt}_1$ on any Nambu-Poisson bracket in dimension $d \geq 3$ are Poisson coboundaries". Obviously, the brute-force computational methods used in the present manuscript will fall short to address (either version of) the conjecture which involves Poisson manifolds of arbitrary finite dimension. Hence, although the present work does provide one data point corroborating the conjecture [for $d = 3$ and γ_3], it unfortunately gives no hint regarding how to address the conjecture in its full generality.

(3) Even if the conjecture (in its more general version outlined above *i.e.* for all wheel generators of $\mathfrak{grt}_1$) was proven to be true, the applications that would derive from such a result are not substantiated enough in the manuscript. In particular, the potential implications of the above conjecture for quantization of Nambu-Poisson structures are not discussed. One would suspect that, if the generalised conjecture holds, then the quantization of Nambu-Poisson structures is unique up to formal diffeomorphisms but this unfortunately is not discussed.

(4) A large part of Section 1 focuses on determining that the presence of tadpoles within the trivialising vector field $X_3^{\gamma_3}$ cannot be avoided. Again, the relevance of such a result is not self-evident and would gain to be further motivated as to justify the great detail of proof displayed to establish it.

(5) Although the introduction alludes to generalisations of the above mentioned results to higher dimension (γ_3 on $d = 4$) or wheel cocycles (γ_5 on $d = 3$), the dedicated section [Section 2] is rather inconclusive, as the only proposition [Proposition 4] merely enumerates cardinalities of sets of graphs whose corresponding value to the reader is not sufficiently justified.

In conclusion, the scope of the present paper is rather small and its main objectives are not enough motivated. Furthermore, even within this restricted scope, it fails to achieve substantial results [as implicitly acknowledged by the authors via the absence of theorems] and the only results displayed are not significant enough as to justify the corresponding over-detailed proofs. In view of the above remarks, it is hard for me to recommend this manuscript as a crucially meaningful addition to the field of deformation quantization or Poisson geometry that would warrant publication in SciPost Physics Proceedings.

4. Questions, comments and suggestions.

(1) One novel contribution of the present manuscript, as contrasted to [2], is to recast some of its results in terms of "micro-graphs calculus", as announced in the title. Recall that the original Kontsevich graphs carry a rich structure (namely one of an operad [6], allowing to define a dg Lie algebra structure on the space of invariants). The term *calculus* employed here suggests that there similarly exists *some* algebraic structure on the space of micro-graphs [as they can in particular be obtained from plugging Nambu-Poisson brackets into vertices of Kontsevich graphs] but this expectation is unfortunately not fullfilled in the manuscript. Furthermore, it comes as somehow unnatural that the definition of micro-graphs involves terrestrial vertices (to be decorated with [non-Casimir] functions) since the original Kontsevich

graphs [3] do not. Naively, this absence of terrestrial vertices is a necessary requirement for this space of graphs to acquire a structure of operad [in contradistinction to the graphs of [4] which do possess terrestrial vertices, but do not carry an operad structure (at least not without modification)]. As such, the term *calculus* may sound too hyperbolic as it seems in this context to refer merely to the manipulation of manifestly $\mathfrak{gl}(d)$-covariant expressions. Perhaps the adoption of a more modest terminology would help to better clarify the scope of the present paper.

(2) Since the focus of this work is on Nambu-Poisson brackets – which do satisfy the Jacobi identity identically – it may be naively expected that there is no need to mod out by the Jacobi identity when computing *e.g.* Poisson differentials [as should be done when working with unspecified Poisson manifolds]. In this respect, the presence of such identically zero terms on the right-hand side of (2) would gain to be further justified [the same can be said for so-called "zero micro-graphs" which by definition identically vanish].

(3) In the resolution of the Open Problem 1, the restriction to micro-graphs possessing at most one tadpole would benefit from being further motivated.

(4) It would perhaps be interesting to not only focus on wheel cocycles but also to evaluate the Kontsevich–Shoikhet cocycle on Nambu-Poisson structures. The latter encodes the obstruction to the existence of loopless quantization of Poisson manifolds [5, 7] hence it could be worth to use this computation to probe the existence of such a loopless quantization for Nambu-Poisson structures.

5. Non-mathematical points (typos, general comments)

(1) The diamond symbol $\diamond$ in equation (2) is undefined.

(2) The integer q in Definition 1 is undefined.

References

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