Referee report for
Mixed permutation symmetry quantum phase transitions...
Authors: Mayourgas et al.

The manuscript should not be published in its current form. The current manuscript contains a number of unclear statements that should be removed or clarified.

1. page 2, second line: What is more "exotic" about N particles distributed in L levels? This is treated at length in many texts, for instance:
   Ping, J., Wang, F., & Chen, J. Q. (2002). Group representation theory for physicists. World Scientific Publishing)

2. page 2, last paragraph of section 1, "we classify the Hamiltonian spectrum and examine..." is in fact false. The authors do NOT in fact classify the spectrum but rather examine a very specific form of the Hamiltonian.

3. The statement "The symmetry will be spontaneously broken in the thermodynamics limit" should be supported by a citation. In particular the author identify the parity $(-1)^N$ as a symmetry of their problem and it's not clear how this symmetry is broken in the example they provide in the submission.

I could go on: the current writing is very sloppy and very unclear. Why would anyone refer to S_{ij} as a "quasi-spin collective operator as it is just by definition a $u(n)$ generator (this is already clear in Eq.(4) of Ref[5]). It is unnecessary to suggest that "The unirreps decomposition can be regarded as a generalization of the Clebsch-Gordan decomposition in U(2)" since this is clearly not $U(2)$. Colloqiual comments like "also called the handgun sector because of its Young frame shape" add nothing to the discussion.

The constant referencing to Ref[5] is inappropriate and misleading. The bibliography can surely afford to include citation to original and/or more recent references pertinent to the problem. For Gelfrand-Tsetlin patterns and matrix elements in this basis, the authors should include references to

1. I. M. Gelfand and M. L. Tsetlin, Dokl. Akad. Nauk SSSR 71, 825 and 1017 (1950), reprinted in I. M. Gelfand, R. A. Minlos, and Z. Ya. Shapiro, Representations of the Rotation and Lorentz Group (Pergamon, New York, 1963).

2. Alex, Arne, et al. "A numerical algorithm for the explicit calculation of SU (N) and SL (N, C) Clebsch–Gordan coefficients." Journal of Mathematical Physics 52.2 (2011): 023507

For $U(3)$ irreps labelled by a one-rowed Young diagram, the expression

$$|h;\alpha,\beta,\gamma\rangle = e^{\beta S_{31}} e^{\alpha S_{21}} e^{\gamma S_{32}} |\mathbf{m}_{\rm hw}\rangle$$

contains redundant parameters since the highest weight state is invariant under a $U(2)$ subgroup. Thus, any one of the exponentials can be eliminated to properly define a CS, but this is not made clear in the text. The authors refer to this implicitly by noting that, for $\mu = 1$, the phase space is 4-dimensional with coordinates α, β so that γ has been eliminated, but this should be clarified. In addition, for irreps labelled by Young diagrams with two or three rows, the phase space is 6-dimensional and I don't understand the relevance of flag manifolds.

The parametrization of U(3) elements given in Eq.(3) is very unusual. At any rate, alternate parametrizations have been considered, some specifically for applications to coherent states. Thus the bibliography should include references to:

1. Klimov, Andrei B., and Hubert de Guise. "General approach to quasi-distribution functions." Journal of Physics A: Mathematical and Theoretical 43.40 (2010): 402001,

2. Klimov, Andrei B., José Luis Romero, and Hubert de Guise. "Generalized SU (2) covariant Wigner functions and some of their applications." Journal of Physics A: Mathematical and Theoretical 50.32 (2017): 323001

3. Nemoto, Kae. "Generalized coherent states for SU (n) systems." Journal of Physics A: Mathematical and General 33.17 (2000): 3493.

It would probably also be useful to add the following paper, specific to SU(3):

1. Byrd, Mark. "Differential geometry on SU (3) with applications to three state systems." Journal of Mathematical Physics 39.11 (1998): 6125-6136.

There are not the only factorizations: one of the more famous is due to Murnaghan, who reprised the work of Hurwitz:

1. Murnaghan, F. D. "On a convenient system of parameters for the unitary group." Proceedings of the National Academy of Sciences of the United States of America (1952): 127-129,

2. Hurwitz, Adolf, Über die Erzeugung der Invarianten durch Integration: Nachrichten von der k. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, 1897, S. 71–90

and was implemented in optical systems by

1. Reck, Michael, et al. "Experimental realization of any discrete unitary operator." Physical Review Letters 73.1 (1994): 58.

There are others and it would therefore seems entirely appropriate for the authors to clarify the benefits of using such an unusual representation to increase the reach and of the results and the readership of the manuscript. Even if their parameterization is only for convenience, a survey of existing parametrization is in my view entirely appropriate.

Beyond the somewhat cosmetic changes mentioned above, there are more fundamental issues. Figure 1 should give some explicit values of the parameters α, β, γ used for plotting purposes, or at least provide some context for the values used in the plots (beyond just sending the reader to Ref[5]), if only for specific points. As currently presented, the figure plots appear independent of α, β, γ. The same applies to Figure 2.

The authors cite Ref[4] to justify the use of coherent states as trial states. However, Ref.[4] introduces coherent states in a different context: to map the Hamiltonian to a phase space and look at the quasi-classical trajectories on that space. This is not quite the same as what is done in this manuscript - or at least the justification is different. Have the authors compared their variational states with the exact eigenstates: is there any reason why the exact eigenstates cannot be computed for the irrep [50,0,0] (which is of dimension 1326) for some values of λ and then compared with the variational states? The same holds for the $N = 30$ states as per figure 2): the irrep [15,15,0] has dimension 136 so it should be a piece of cake to get the lowest eigenvalue and ground state, and compare with the variational state to justify the use of this ansatz.

The N-fold tensor product of the fundamental representation of $U(3)$ (or $U(n)$ for that matter) is easily decomposed using Schur-Weyl duality. In particular, there will be multiple copies of all representations except the fully symmetric and the fully antisymmetric (if the latter appears at all). I find the current exposition incomplete in that there is no discussion on the effect of these multiple copies of the same representation. For instance the irrep corresponding to the Young diagram [N-1,1,0] appears N-1 times. Have these $N-1$ copies been accounted for, and if so how (in particular when comparing results with the fully symmetric representation [N,0,0], which occurs once)? Does this affect the shape of the curves in Figs. 1 and 2? What does permutation symmetry mean in this subspace when $N \to \infty$? How is the limit to be taken for states in subspaces of mixed symmetry?

On balance, it seems that this submission is just a watered-down digest of Ref.[5]. I realize this is a conference proceedings and that novelty is not necessarily required, but if the authors are going to constantly refer to their own work maybe their contribution should not go in the proceedings.