Report on *t' Hooft lines of ADE type and topological quivers*

In his seminal work 'Supersymmetric gauge theory and the Yangian' Costello proposed a connection between integrable spin chains and an unusual mixed-topological theory now known as 4d Chern-Simons (4dCS). In particular, the R-matrices of spin chains are realised as the vacuum expectations of crossing Wilson lines, and concatenating them gives transfer matrices. The Yang-Baxter equation and RTT relations then follow from diffeomorphism invariance of 4dCS in the topological direction.

For a number of years a central ingredient was missing from this narrative: the Baxter Q-operator. This omission was resolved by the paper of Costello, Gaiotto and Yagi: 'Q-operators are t' Hooft lines.' Therein the authors demonstrate how the Baxter Q-operators for $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_{2n}(\mathbb{C})$, $\mathrm{E}_6(\mathbb{C})$ and $\mathrm{E}_7(\mathbb{C})$ can be realised in 4dCS. They are obtained by concatenating more primitive objects, L-operators, which arise as vacuum expectations of crossing Wilson and t' Hooft lines. The latter are taken to have magnetic charge equal to a miniscule coweight of the adjoint form of the group. As part of this analysis, the authors obtained oscillator constructions of these L-operators when the Wilson line is in the fundamental of $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{SO}_{2n}(\mathbb{C})$, matching known descriptions in the literature. This procedure was also sketched in the case of $\mathrm{E}_6(\mathbb{C})$, in which case the resulting oscillator construction was novel.

In this paper, the above procedure is applied for various choices of representations for the Wilson line R and miniscule coweights μ to obtain oscillator realisations for L-operators. As the miniscule coweights are a subset of the fundamental coweights, they can be labelled by a Dynkin index. The particular cases considered are:

- $G = \mathrm{SL}_n(\mathbb{C})$: $R = \mathbf{n}$ and $\mu = \mu_k$ for $k = 1, \dots, n-1$.
- $G = \mathrm{SO}_{2n}(\mathbb{C})$: $R = \mathbf{2n}$ and $\mu = \mu_1$ (the vector representation), $\mu = \mu_n$ (one of the two spinor representations).
- $G = \mathrm{E}_6(\mathbb{C})$: $R = \mathbf{27}$ and $\mu = \mu_1$.
- $G = \mathrm{E}_7(\mathbb{C})$: $R = \mathbf{56}$ and $\mu = \mu_1$.

It's my understanding that all of these L-operators have appeared in the literature previously, although the exceptional cases only recently in the related work 'On exceptional 't Hooft lines in 4D-Chern-Simons theory' by a subset of the authors. Nevertheless, the present work provides a thorough exposition of their construction.

The remainder of this paper concerns a procedure for assigning quiver diagrams, termed topological gauge quivers, to triples (G, R, μ) for G the gauge group, R the G representation of the Wilson line and μ a miniscule coweight determining the magnetic charge of the t' Hooft line. These diagrams have a formal similarity to those determining supersymmetric quiver gauge theories. The topological gauge quiver provides a succinct way of encoding much of the data relevant in the construction of the corresponding L-operator. The authors illustrate these quivers for the choices of G and μ listed above, and for a wider class of representations R. For example, in the case of $G = \mathrm{SL}_n(\mathbb{C})$ and $\mu = \mu_k$ for $k = 1 \ldots, n-1$ they find the corresponding topological gauge quivers for $R = \mathbf{n}, S^2\mathbf{n}$, $S^3\mathbf{n}$, $\Lambda^m\mathbf{n}$ (for $m \geq 2$) and $\mathbf{ad}$. In the case of $E_7(\mathbb{C})$ the authors note that sequentially specifying miniscule coweights in the Levi subalgebra generates a chain of ADE type algebras. In this sense 4dCS for $G = E_7(\mathbb{C})$ unifies many of the other examples.

In order to recommend publication, I feel that the following points would need to be addressed:

- The procedure for assigning a topological gauge quiver to the data (G, R, μ) is not transparent to me. The choice of miniscule coweight distinguishes a (reductive) subalgebra $\mathfrak{l} \subset \mathfrak{g}$, and the representation R then decomposes into irreducibles. Am I correct in thinking that the nodes of the quiver are labelled by these irreducibles? The links between nodes are then labelled by products of Weyl algebra generators. How are these determined? An explanation of exactly how a topological gauge quiver is constructed would help clarify the paper.

- Is it possible to explicitly recover an L-operator from the corresponding quiver? To my knowledge oscillator L-operators are not known for all choices (G, R, μ) for which topological gauge quivers are constructed. The quiver description would be better motivated if it could be exploited to generate a novel L-operator.

- In the work of Costello, Gaiotto and Yagi it's proposed that phase spacse of t' Hooft lines can be identified with the Coulomb branches of certain ADE quiver gauge theories. These can naturally be quantized by Ω deformation. The groups and flavours assigned to the quiver are related to magnetic charges of the t' Hooft lines at $z = 0, \infty$. Is there any connection between these quiver gauge theories and the topological gauge quivers considered in the manuscript?

- In figure 30 topological gauge quivers are illustrated for R the adjoints of $\mathrm{SO}_{2n}(\mathbb{C}), \mathrm{E}_6(\mathbb{C}), E_7(\mathbb{C})$. However, the adjoint representation does not lift to the corresponding Yangian in these examples, and so there are no corresponding Wilson lines in 4dCS. Are the associated L-operators to be interpreted semi-classically? If so, distinguishing in which cases semi-classical vs quantum L-operators are being constructed would be useful.

- Explicit L-matrices appear in equations (3.61), (3.74), (4.101), (5.168, (6.204) and (7.242) (and the subsequent equations). I understand these examples have all appeared in the literature elsewhere, although in the exceptional cases only in the related paper 'On exceptional 't Hooft lines in 4D-Chern-Simons theory'. If so, it would be useful to include references to the original works in which they appear. If they are novel, it would be worth stating this explicitly.

I also have some further minor comments:

- The equation of motion of 4dCS (2.4) is not correct. It should read

$$\mathrm{d}z \wedge (\mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}) = 0\,. \qquad (1)$$

  In particular this is not topological in the $\mathbb{CP}^1$ direction.

- In equation (2.14) both instances of z^{μ_R} should be replaced by μ_R.

- In equation (2.24) does sl_1 refer to $\mathbb{Z}_n$ or $\mathbb{C}$? Presumably the latter?

- I understand that the second column of table (2.26) lists representations of $sl_1 \oplus sl_n$. What is the representation $\bar{\mathbf{1}}$? On a related note, in the third row representations appear in pairs, e.g., $\mathbf{F}_{-1/N}\bar{\mathbf{1}}_{1/N-1}$. Is there an implicit tensor product here? If so surely we can add the charges and remove the $\bar{\mathbf{1}}$ factor. (This is repeated elsewhere in the manuscript.)

- In equation (3.60) the combination $Y\rho_{\bar{1}}$ appears, but this vanishes by equation (2.37). I think Y should appear on the r.h.s. of $\rho_{\bar{1}}$.

- In figure 11 the red arrow and blue arrows connecting the $\underline{k(N-k)}$ and l_{μ_k} nodes appear to be backwards. The charges do not match up.

If these points are addressed I believe the manuscript would meet the acceptance criteria.