Report: Crossed product algebras and generalized entropy for subregions

Dear Editor,

This manuscript discusses a generalization of a recent discussion of crossed products in gravity algebras to the Rindler space. The authors claim "*the crossed product construction represents a refinement of Haag's assignment of nets of observable algebras to spacetime regions by providing a natural construction of a type II factor*". I have several conceptual comments and questions about the draft.

1. Page 1: *While any open spacetime region may be assigned a von Neumann algebra A(O), imposing further axioms on the assignment (1.1) leads to relations between the algebras defined on different regions*

   I find this statement misleading. We only associate C*-algebras to open sets in spacetime. I do not know of any ways to assign a von Neumann algebra to an arbitrary open set in spacetime without taking the double commutant that, in local QFT with timeslice axiom, enlarges the region to its causal completion.

   As the authors elaborate: For a topologically trivial region, in many quantum field theories, in the standard KMS state, one proves Haag's duality. Taking the double commutant results in von Neumann algebras that are associated with double cones, or causally complete regions. For example, see the "time-like tube theorem".

2. Page 2: *These infinities arise from the lack of a finite trace on the algebras of observables, thereby obstructing the existence of density states localized to the region. This is due to the fact that the algebra of operators in type III theories does not admit a tensor factorization, in contrast to type I factors appearing in quantum mechanics*

   This statement is correct but misleading. Note that the issue of the existence of trace and tensor factorization are not to be confused. Type II_1 algebras have finite trace but still do not admit a reducible presentation on $B(\mathcal{H})$, required for tensor factorization.

3. Page 3: *In stark contrast to type III algebras, type II algebras do admit a well-defined trace, and hence a meaningful definition of von Neumann entropy for density states.*

   It is important to mention that the von Neumann entropy of type II algebras are still different in nature from type I entropy. It is crucial to remember that the type II_∞ entropy is not derived from counting states, as opposed to the case of type I algebras.

4. Page 3: *The issue with this in the type III_1 algebra is two-fold: there is no finite trace with which to normalize the density states, and there is no operator in the algebra of the subregion that generates them*

   I suggest changing *finite trace* to *trace*. Whether the trace is finite or not is not relevant for the discussion. Note that the trace of type I_∞ is not finite. I do not understand what is meant by *there is no operator in the algebra of the subregion that generates them.*

5. Page 5: *Physically, inner automorphisms are simply unitary transformations.*

   For clarity, I suggest adding *unitary transformations that belong to the algebra.* Because outer automorphisms are also unitary transformations. In fact, any automorphism of a von Neumann algebra can always be realized as a unitary transformation on the GNS Hilbert space.

6. Page 6: *it is a standard result in the theory of operator algebras that the crossed product of a type III_1 algebra with an outer automorphism is a von Neumann algebra of type II_∞.*

   I do not think this is a correct statement. What the authors mean is the crossed product with the modular group. As a counter example, consider a theory with a finite group as global charge, e.g. $\mathbb{Z}_2$. The crossed product of the local algebra of two disjoint regions with the $\mathbb{Z}_2$ group of outer automorphisms corresponding to intertwiners is still type III_1.

7. Page 6: *In general, we will take it to be the Hamiltonian of the commutant $\mathfrak{A}'$.*

   I do not understand what the authors means by the Hamiltonian for the commutant algebra. Note that if $\mathfrak{A}$ is type III_1 then $\mathfrak{A}'$ is also type III_1. Note that in the case of holography, the author of [27] had a definition of this operator. But that applies only to the holographic setups. This is one of my main criticisms of this work.

8. Equation 3.8: I am not sure what this equation means. In the case of holography, there was a large N parameter and we had a boundary definition of this operator. Here, the use of G_N and the equation 3.13 seem ad hoc to me.

9. Section 4: As far as I can understand the authors point out the observation that Haag's duality favors entanglement wedge over causal wedge. An observation that appears and has been discussed in the literature before; for example see 2008.04810. This is often interpreted as a non-uniqueness of the choice of von Neumann algebras one can associate to GFF algebras, for example see 2210.00013. There are correspondingly two modular flows, each corresponding to one choice of von Neumann algebras. I fail to see how the crossed product construction "*represents a refinement of Haag's assignment of nets of observable algebras to spacetime regions by providing a natural construction of a type II factor*" as claimed by the authors.

The manuscript contains interesting and important discussions. However, I do not recommend it for publication in its current form.