Title: Lack of near-sightedness principle in non-Hermitian

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The paper describes that introducing an "impurty" in the hopping can eliminate the so-called the non-Hermitian skin effect. The authors use a transfer-matrix formulation to show the evidence numerically.

The authors do not consider a topological *insulator* but demand the non-Hermitian system to obey the "near-sightedness principle." The numerical evidence that the authors found is actually argued analytically as I show below. I thereby find the authors' claim that the non-Hermitian skin effect is not a topological phenomenon quite unfair. I therefore do not recommend its publication.

First of all, the authors define the "near-sightedness principle" as the robustness of the insulator, and particularly the topological insulator, against local perturbations. This is because the bulk states of the insulator are localized in real space. The authors incorrectly extend only the part "topological" to non-Hermitian systems, ignoring the part "insulator" and mistakenly conclude that the non-Hermitian skin effect is not topological because it is not robust against a local change. Since the authors' model originally does not contain any randomness or any other effects to localize bulk states, it is not an insulator and surely does *not* obey the "near-sightedness principle." If the authors would introduce the randomness as the original Hatano-Nelson model, the conclusion would have been different.

Next, the non-Hermitian skin effect is analytically explained by the introduction of the imaginary gauge transformation, For the Hatano-Nelson model under the open boundary condition,

$$H_{NH} = \sum_{x=1}^{L-1} (t_R \left|x+1\right\rangle\!\left\langle x\right| + t_L \left|x\right\rangle\!\left\langle x+1\right|), \tag{R1}$$

consider the gauge transformation

$$|x\rangle\!\rangle = \mathrm{e}^{hx} \left|x\right\rangle, \quad \langle\!\langle x| = \mathrm{e}^{-hx} \left\langle x\right|, \tag{R2}$$

where h is defined in

$$\mathrm{e}^{2h} = \frac{t_R}{t_L}. \tag{R3}$$

We can thereby transform the original non-Hermitian Hamiltonian (R1) to the Hermitian one:

$$H_H = \sum_{x=1}^{L-1} (t \; |x+1\rangle\!\rangle\langle\!\langle x| + t \; |x\rangle\!\rangle\langle\!\langle x+1|), \tag{R4}$$

where $t := t_R e^{-h} = t_L e^{h}$. Equivalently, in a matrix formulation, the original Hamiltonian (R1) given by

$$H_{NH} = H(h) = \begin{pmatrix} 0 & te^{-h} & 0 & 0 & 0 \\ te^{h} & 0 & te^{-h} & 0 & 0 \\ 0 & te^{h} & 0 & te^{-h} & 0 \\ 0 & 0 & te^{h} & 0 & te^{-h} \\ 0 & 0 & 0 & te^{h} & 0 \end{pmatrix} \tag{R5}$$

for $L = 5$, for example, is Hermitized by the similarity transformation

$$S = \begin{pmatrix} e^{h} & & & & \\ & e^{2h} & & & \\ & & e^{3h} & & \\ & & & e^{4h} & \\ & & & & e^{5h} \end{pmatrix} \tag{R6}$$

as in

$$S^{-1} H(h) S = H(0) = H_H. \tag{R7}$$

The eigenvalue spectrum of the original Hamiltonian (R1) is hence equal to that of the Hermitian Hamiltonian (R4); in other words, the spectrum stays real for any non-Hermiticity as long as $t_R t_L > 0$.

On the other hand, the eigenfunctions are transformed accordingly. The right-eigenfunction of the original Hamiltonian (R1) is skewed from the eigenfunction of the Hermitian Hamiltonian (R4) as in

$$\psi_{NH}^{R}(x) = e^{hx} \psi_H(x) \tag{R8}$$

which grows exponentially towards the right edge, resulting in the non-Hermitian skin effect. Similarly, the left-eigenfunction is skewed

$$\psi_{NH}^{L}(x) = e^{-hx} \psi_H(x) \tag{R9}$$

which grows exponentially to the left. This is the standard argument of the non-Hermitian skin effect.

Based on this argument, we can come up with an infinite number of non-Hermitian Hamiltonians with the common eigenspectrum. For example, by non-Hermitizing the Hermitian Hamiltonian (R4) with a similarity transformation

$$S_{\text{imp}} = \begin{pmatrix} e^{h} & & & & \\ & e^{2h} & & & \\ & & e^{2h-h_{\text{imp}}} & & \\ & & & e^{3h-h_{\text{imp}}} & \\ & & & & e^{4h-h_{\text{imp}}} \end{pmatrix} \tag{R10}$$

as in

$$H_{NH\mathrm{imp}} = S_{\mathrm{imp}} H_H (S_{\mathrm{imp}})^{-1}, \tag{R11}$$

we have the authors' Hamiltonian (4), or equivalently

$$H_{NH\mathrm{imp}} = \begin{pmatrix} & t\mathrm{e}^{-h} & & & \\ t\mathrm{e}^{h} & & t\mathrm{e}^{h_{\mathrm{imp}}} & & \\ & t\mathrm{e}^{-h_{\mathrm{imp}}} & & t\mathrm{e}^{-h} & \\ & & t\mathrm{e}^{h} & & t\mathrm{e}^{-h} \\ & & & t\mathrm{e}^{h} & \end{pmatrix} \tag{R12}$$

The right-eigenvector for the resulting non-Hermitian Hamiltonian (R12) is accordingly skewed as

$$\vec{\psi}^{R}_{NH\mathrm{imp}} = \begin{pmatrix} \mathrm{e}^{h} & \psi_H(1) \\ \mathrm{e}^{2h} & \psi_H(2) \\ \mathrm{e}^{2h-h_{\mathrm{imp}}} & \psi_H(3) \\ \mathrm{e}^{3h-h_{\mathrm{imp}}} & \psi_H(4) \\ \mathrm{e}^{4h-h_{\mathrm{imp}}} & \psi_H(5) \end{pmatrix} \tag{R13}$$

It surely decreases exponentially at the right edge if $(L-1)h < h_{\mathrm{imp}}$ but increases exponentially up to the hopping impurity. Notice that the conclusion would be different if the eigenvector $\psi_H(x)$ for the Hermitian Hamiltonian were localized due to a random potential, and hence the system were indeed an insulator.

Finally, the reference list lacks some important papers. Upon introducing the Hatano-Nelson Hamiltonian, the authors do not cite the original Hatano-and-Nelson papers:

- N. Hatano and D.R. Nelson, Phys. Rev. Lett. **77**, 570 (1996)
- N. Hatano and D.R. Nelson, Phys. Rev. B **56**, 8651 (1997)

The critical papers on the non-Hermitian skin effect and those on the topological property of non-Hermitian systems are lacking too.

- S. Yao and Z. Wang, Phys. Rev. Lett **121**, 086803 (2018)
- Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X **8**, 031079 (2018)
- K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Phys. Rev. X **9**, 041015 (2019)

Also, the review paper [4] lacks the names of the authors.