Referee report on “Exact first-order effect of interactions on the ground-state energy of harmonically-confined fermions” by Pierre Le Doussal, Naftali R. Smith, Nathan Argaman

The authors perturbatively compute the first order correction E_N^{(1)} to the ground state energy E_N^{(0)} in certain many body quantum systems. The setup consists of N spinless Fermions in d=1,2 and 3 dimensions, confined by a harmonic potential, under the condition that all single energy levels are filled up to N. This provides a relation between the harmonic oscillator Fermi energy parametrized by an integer M and N, Eq. (5). The weak interaction is assumed to be translational invariant and of power-law type, \sim \epsilon/r^n, under the condition 0<n<d+2 for convergence. In the case of the Coulomb interaction in three dimensions with n=1, the asymptotic expansion of E_N^{(1)} is worked out up to several orders in fractional powers of N, including logarithmic corrections. It is based on the exact results for finite N, using generating functions. The leading and next-to-leading order are compared with numerical simulations, and the leading order is rederived from a semi-classical expansion in section 4. Several appendices with technical details round off the paper with an extensive introduction and discussion section.

The paper is very well written and contains an impressive amount of detailed results, some of which hinting beyond the presented scheme. I have not been able to check all results in detail for correctness, in particular all the appendices, but the results seem to me sound and well founded, also in comparison to the numerics. I am not an expert in the detailed literature beyond leading order, but to me the results seem novel and certainly justify publication, also in comparison to the (re)derivation of existing results from semi-classics.

Below follows a longer list of small corrections, typos and optional questions. It will not be difficult for the authors to take them into account, after which I strongly recommend publication in SciPost.

List of comments:

1. Whenever the authors speak of Coulomb interaction (n=1), they should add “three-dimensional” Coulomb interaction, starting in line 6 of the abstract. This is because in one respectively two dimensions it is linear respectively logarithmic, as you may add somewhere.
2. An extensive account is given for the expansion of the ground state energy of neutral atoms in the atomic number Z. From Section 2 onwards the role of the expansion parameter is played by N. However, no comment is made relating the two, please do.
3. Page 2 line 7: its mean density is not spatially uniform. Add ”in general”, see e.g. [20] where it is.
4. End of section 1.1: perhaps add “temperature” to the list, see e.g. [19,23].
5. It should be mentioned that the power law interaction in (4) is also called Riesz interaction, for d>n, also after Eq. (23). For the Riesz gas there is quite some recent mathematical literature, see e.g. [A] and references therein.
6. Typos: in Eq. (9) the exponent should read – {\bf x}^2/2 (or each factor should be pulled under the product).
   Likewise, the two terms x^2+y^2 in the exponent of Eq. (30) should be in bold
   Why are absolute values and complex conjugates introduced in Eqs. (10) and (11)? Although true in general, it seems to me that for the harmonic oscillator (HO) treated

here, all wave functions are real, cf. (9). Therefore, the kernel in (11) and (31) will be automatically symmetric. Please comments on this.

7. It might be useful for the reader to state that that Eq. (39) is nothing but the Laplace transform, with s={\bf b}^2/2, also in view of the comment about long range potentials, please add some details here.
8. I would generally give a reference to standard tables of integrals when it comes to identities such as Eq. (43), especially in view of the analytic continuation.
   This would also apply to standard sums in (6) or (7).
   Although one can check these e.g. with the software Mathematica, I am not satisfied with such a reference in general.
9. After Eq. (59) (or even (51)) it might be useful to state the definition of these hypergeometric functions, also to see why *3F*2 truncates I this case
10. At first I did not understand the logic of the asymptotic expansion between eqs. (61) and (68). Please try to explain this better as it is of crucial importance for the paper.
    As far as I understand there are two claims made here:
    I) An Ansatz can be made, such that the asymptotic expansion in \eta = 1-z of (65) and (66) agree to a certain order (this will lead to a linear set of equations for the coefficients in (66), hence to the claim for the uniqueness – this cannot be proven with Mathematica but by linear algebra statements).
    II) The expansion in (67) can be read off by (asymptotically0 identifying M^s in Li_{-s}(z) from (63)

Question: is this related to a well-known mathematical procedure, like Pade approximation, where the error could be made explicit?
Notation: why is the notation in the error term in (67) change in (68)? Both should read o(M^{-1/2} because of potential logarithms, and not O(M^{-3/2}), correct?

11. Page 21 line 2 after (100): reference [48] does not provide a definition of the complete elliptic integrals E and K, please use NIST Handbook or Gradshteyn Rhizknik for example.
12. Do you have an intuition why n=1 and n=3 agree to this order in (114)?
13. Appendix A: [55] does not quite provide the expansion (116), it has to be combined with the expansion of (-log(1-x))^{-s-1}, please give some more details.
    In (118) a factor 1/n! seems to be missing in the first sum.
14. Appendix B.2: in (136) please use the more common notation 1/sin(x) instead of cosecans csc(x), or define it
15. Figs. 4 and 5.: do you have an intuition why in this particular case only from such a large value of M between 50-100 onwards the approximation works well? This is very much in contrast to few per cent for all other cases, due to the cancellation at d=n?

References:

[A] Lewin, M. (2022). Coulomb and Riesz gases: The known and the unknown. *Journal of Mathematical Physics*, *63*(6).