In this work, the authors present a systematic and rigorous study of the electronic properties of insulators, focusing on their behavior as described by the optical sum rule. The sum-rule inequalities reveal the relationship among the localization gap (E_L), the Penn gap (E_P), and the optical gap (E_G), which serve as the foundation for determining lower and upper bounds of the localization length (l). In the second part of the study, single-electron and two-electron (van der Waals dimer model) systems are solved analytically, demonstrating agreement with the inequalities and localization results. The localization lengths of various materials are estimated, showing reasonable consistency with data obtained from first-principles calculations.

I find this work well-written and clearly explained. While the localization length estimation relies on experimental data and may predict a relatively broad range compared to first-principles methods, I believe this work should be published without significant changes.

Below I bring some of my questions.

Questions:

1. As the Drude weight is introduced at the beginning of the paper, is it also possible to characterize it using optical absorption equations? In other words, can this methodology be extended to metallic systems without a fundamental gap in the DC limit?

2. In Section 2, paragraph 2, there is an assumption: "For simplicity, we will assume cubic symmetry or higher so that...". I wonder if this assumption is overly simplified when considering the localization length in real materials. Would the results hold for systems with lower symmetry?

3. The Van der Waals dimer model, where the two-electron system is analyzed in detail, provides a good starting point for studying many-body physics. However, since the correlation function is not considered and the resulting localization length does not differ significantly, I am curious about the rationale for introducing this many-body model. Does it offer additional insights into specific material properties, especially given that the results seem to saturate in the long-wavelength limit?

4. In the comparative analysis shown in Figure 4, only tetrahedrally-coordinated materials are included. Is it possible to extend the comparison to other material systems mentioned in the article, such as diamond, zincblende, or rocksalt alkali halides? Furthermore, do the differences between first-principles and the presented method depend on crystal symmetry? For example, do tetrahedrally-coordinated materials exhibit better agreement due to their higher symmetry?

5. Following the previous question, if E_G represents both the band structure energy gap and the optical gap (using the same notation), is it possible to extract E_L and E_P directly from the band structure? If so, would experimental bounds no longer be necessary?