Report on "Critical dynamics and cyclic memory retrieval in non-reciprocal Hopfield networks" submitted to SciPost (Physics) by Xue, Maghrebi, Mial and Piermarocchi (2025).

This paper investigates the dynamics of an attractor neural network in the Hebbian family of Hopfield models, at work with solely two patterns.
Due to this constraint, a whole analytical treatment of the dynamics (whose inspection is in general of prohibitive difficulty) is feasible and carries interesting insights from a physical standpoint.
The Authors use a non-symmetric generalized Hebbian coupling (a very clever one where the transition from state A to state B is twice forced by the way the write the asymmetric term in \lambda- in eq.(3)), hence they rule out the standard statistical mechanical approach (that would be useless in any case since the patterns are just two), rather they tackle the problem from a dynamical perspective.
A huge inspection on the dynamics is then achievable, in particular the Authors focus on the critical one (as fairly standard) at various level of description (from simplest mean-field, to including noise, toward a full master equation approach and even a Liouvillean formulation), making this paper a nice example of the richness insite these Hebbian models.
Beyond these general arguments, the Author salso work out explicitly the decorrelation time finding the scalings analytically and confirming them computationally.
Overall the paper is well written and it is interesting to read hence I would be tempted to suggest acceptance, yet there is a number of questions/points to be deepened IMHO, hence -by now- I suggest to the Editor “Minor Revisions” such that, should the Authors write a revised version and a report, I will be happy to play again as Referee for this paper.

Hereafter I share my comments and questions following page after page the original manuscript.

-The portraits in the plane m_1 vs m_2 shown in Fig.1 are interesting as these are quite standard from a dynamical system perspective yet they are not the standard way to describe Hopfield nets, hence I particularly enjoyed. Yet I did not understand, if I focus on the last two plots (the last row, with plots “e” and “f”) what is the center (m_1=m_2=0)? Furthermore, you have four sinks because you store the two patterns and the two gauge-related patterns? (i.e. \xi^1 and -\xi^1 for pattern 1)?

-at pag. 7 there is a first written line than there is an empty space...

-In sec. 3.1 I do not entirely understand why in eq. 25 the r.h.s is a free energy and not a standard energy function. It seems quite a Langevin equation to me, hence I would expect to read H and not F. Furthermore, if that is a free energy, it depends on the noise (i.e. the temperature) as F= E – TS, where E is the energy, T the temperature and S the entropy. If I look at the “free energy” (27) I don’t recognize nor the temperature neither the entropy contributions. Perphaps I can be wrong but as I did not understand, this may happen also to other readers, hence an explanation would be helpful (or simply the substitution F \to H would solve the impasse).

-the same problem is in eq. (33) where I recognize an energy but barely a free energy…

-In Sec. 3.3 the question of the Goldstone mode is interesting but subtle (I already seen this in Andrea Cavagna’s papers): I would add a citation to a paper that the Authors think relevant for understanding for the general reader not aware of a Goldstone mode…

-After eq. 61, the Authors cite (using their bibliography) [36,37] to highlight research on networks without self-interactions but those papers where on a slighlty different problem: Personnaz and coworkers were investigating unlearning protocols in Hebbian nets, while Kanter and Sompolinsky worked out the statistical mechanical version of the Kohonen net, yet these two papers are deeply linked as the (correct) unlearning scheme for the Hopfield network allows the model to collapse to the Kanter-Sompolinsky one as explained in

[x] Alberto Fachechi,et al. "Dreaming neural networks: forgetting spurious memories and reinforcing pure ones." *Neural Networks* 112 (2019): 24-40.

While, recently, David Saad spent a paper to address the role of self-interactions, that is

[x] David Saad et al. "High storage capacity in the Hopfield model with auto-interactions—stability analysis." *Journal of Physics A: Mathematical and Theoretical* 50.46 (2017): 465001.

But there is a matter of taste underlying hence this is a very minor point…
Further, along the same line, I also point out that both the research groups on neural nets in Rome and Tokyo are inspecting very similar researchlines, see e.g.

[x] Kang, Louis, and Taro Toyoizumi. "Hopfield-like network with complementary encodings of memories." *Physical Review E* 108.5 (2023): 054410.
[x] Agliari, Elena, et al. "Generalized hetero-associative neural networks." *Journal of Statistical Mechanics: Theory and Experiment* 2025.1 (2025): 013302.
[x] Agliari, Elena, et al. "Networks of neural networks: more is different." *arXiv preprint arXiv:2501.16789* (2025).

Also, as a last point regarding the bibliography, I think that a very early PNAS by Amit -where the idea of coupling \xi^{\mu} to \xi^{\mu+1} was introduced- is missing: just in case, it is:

[x] Dani Amit. "Neural networks counting chimes." *Proceedings of the National Academy of Sciences* 85.7 (1988): 2141-2145.

-finally, a question I'd like to ask is about the stability of the painted picture when the number of patterns is minimally increased, but I am not sure we can save O(1) of the complete scenario the Authors wrote…