Referee Report on [SciPost-02505.00059v1]

In this manuscript, the authors present a method to systematically generate the (exponentially large number of) Feynman diagrams for the self-energy of the single polaron, starting from its expression in terms of the vertex function. The reference self-energy is the Self-Consistent Born Approximation (SCBA), and vertex corrections are then introduced iteratively with the aim of asymptotically reconstructing the exact self-energy.

While the work contains some non-trivial insights into the structure of perturbation theory, my overall assessment is that it is not yet suitable for publication in *SciPost*. The paper is best viewed as a preliminary contribution which could be the basis for a more comprehensive and refined treatment. My main concerns are summarized below and expanded upon later in the report:

- The approach is highly specific to the diagrammatic structure of the single-polaron problem. It is not at all straightforward to see how it could be generalized to more complex systems, such as those with finite particle densities or boson propagator renormalization.
- The impact of this method on the sign problem encountered in diagrammatic Monte Carlo (DMC) methods is not adequately discussed. In particular, the origins of the sign problem in the single-polaron case differ substantially from those in many-fermion systems, and this distinction is not clearly addressed.
- There is a lack of quantitative comparison with existing DMC results for the same problem, which would be essential to evaluate the accuracy and utility of the proposed method.
- The manuscript suffers from presentation issues that hinder readability and accessibility, including a lack of clarity in mathematical notation and the sequence of conceptual exposition.

Despite these concerns, the authors' use of Dyck paths and continued fractions offers an elegant combinatorial framework that could inspire future developments. Below, I provide more detailed comments and specific suggestions for improvement.

Detailed Comments

- The authors exploit a mathematical equivalence between Dyck paths and continued fractions. They relate SCBA diagrams to non-crossing diagrams and introduce vertex corrections by dressing the bare vertices of lower-order diagrams. While this is correctly implemented, it resembles standard techniques for including vertex corrections and does not clearly stand out as a novel contribution.

- The claimed computational advantage of generating diagrams via Dyck paths is limited, as the actual evaluation still requires summing over momenta, as shown in Eq. (5). Thus, the bottleneck remains in the numerical evaluation of vertex-corrected diagrams, even in the single-polaron case.

- The statement that the method inherently avoids the sign problem is not substantiated in the manuscript. This is a significant claim, and if true, would merit prominent discussion and quantitative evidence. As it stands, the argument is unconvincing.
- The possibility of extending this method to finite densities or to the renormalization of the boson propagator is mentioned briefly, but without concrete justification. In those more complex regimes, the diagrammatic structure differs substantially, and the exponential growth of diagrams—absent in the single-polaron limit—presents challenges that are not addressed.

Technical and Presentation Issues

- The acronym SCBO is non-standard and confusing. The well-established term in the literature is SCBA (Self-Consistent Born Approximation), and this should be used throughout.
- Eq. (9): The use of vertical bars in the continued fraction is unconventional and confusing. The notation should be clarified, or ideally revised to match standard usage. The subsequent sentence $c_0 z^2=g^2 G_0(\omega-\omega_0)$ is rather confusing here. The exact meaning of z^2 is indeed expressed in sec. 4.
- Eq. (11): The use of multidimensional indices $\vec{m}$ and the set N^{h+1}_0 requires clearer explanation and explicit definition.
- The introduction of Dyck paths precedes their physical interpretation in terms of Feynman diagrams. This sequence is conceptually disorienting. I recommend either introducing the formalism later or providing physical context earlier.
- Fig. 13: The meaning of the two dashed lines should be clarified in the caption or the main text.
- Eq. (28): The right-hand side should read G_0, not G.
- Eq. (30): The Ward identity is non-trivial in this context and should be discussed more carefully.
- Section 5.2: The discussion around Eq. (32) does not add much beyond standard perturbation theory. The connection to the derivative of the self-energy with respect to an external field could be better emphasized.
- The term “irreducible self-energy diagrams” must be clearly defined. Typically, the self-energy is composed of one-particle-irreducible (1PI) diagrams.

In other words all self-energy diagrams are 1PI. If a different notion of reducibility is used, it should be stated explicitly.

Summary of Strengths

- The use of continued fractions and Dyck paths introduces a potentially elegant organizational principle for self-energy diagrams.
- The authors provide a combinatorial framework that could in principle be useful for algorithmic generation of diagrammatic contributions.

Summary of Weaknesses

- The scope is restricted to the single-polaron case, with no clear path to generalization.
- The relevance of the method to the sign problem is speculative and unsupported by evidence.
- Lack of benchmarking against established DMC methods in the same context.
- Several technical and presentational issues that hinder clarity and accessibility.

Assessment of SciPost Criteria

- **Clarity and intelligibility**: *Not fully matched.* The presentation requires substantial revision for clarity.
- **Reproducibility**: *Matched.*
- **Citations and context**: *Matched.*
- **Conclusion and perspective**: *Not adequately matched.* The broader implications and limitations are not sufficiently discussed.
- **Introduction and abstract**: *Partially matched.*
- **Reproducibility resources**: *N/A.*

In conclusion, while the paper offers interesting combinatorial insights, I believe it requires significant revision and additional analysis before it could be considered for publication in *SciPost*. I encourage the authors to develop the method further, including a critical evaluation of its computational advantages and limitations, and to provide concrete applications or comparisons with existing approaches.